Dual boundary complexes of Betti moduli spaces

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Outline:

1. Dual boundary complexes and the geometric P=W conj

2. Legendrian knots and constructible sheaves

3. Betti moduli space via augmentations

(4. A cell decomposition of the Betti moduli space)

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1. Dual boundary complexes and the geometric P=W conj.

C = Riemann surface (with punctures)

$G = GL_n(\mathbb{C})$ (or more generally, linear reductive)

Nonabelian Hodge correspondence (NAH) $\Rightarrow$

$\Phi: \mathcal{M}_{Dol} \xrightarrow{\sim} \mathcal{M}_B$ real analytic isomorphism

(Not algebraic)

$\mathcal{M}_{Dol}$:= Dolbeault moduli space of G-Higgs bdles over C ...
(stable, vanishing chern class, ...)

$\mathcal{M}_B$:= Betti moduli space of G-local systems over C ...
(irreducible, ...)

$\Rightarrow \Phi^*: H^*(\mathcal{M}_B) \cong H^*(\mathcal{M}_{Dol})$.

but does not preserve the mixed Hodge structures (MHS) cohomological P=W conjecture (de Cataldo–Hausel–Migliorini 12).

NAH exchanges

the weight filtration on $H^*(M_B)$ (algebraic geometry)
(W)

and

the perverse-Leray filtration on $H^*(M_{Dol})$ wrt
(P)

the Hitchin map $h: M_{Dol} \to A$. (topology)

what do we know?

- deCHM 12: true for rank 2, any genus
- deCMS 19: true for any rank, genus 2.

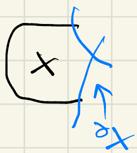
Q: Geometric interpretation?

$\rightsquigarrow$ Geometric P=W conjecture: concerns NAH at infinity.

Dual boundary complex

$X (= M_B)$ smooth affine $\rightsquigarrow$ log compactification $\overline{X}$ with

∂X = s.n.c. boundary divisor.



Blowing up $\Rightarrow$ may assume the intersections of irreducible components of ∂X are connected.

Say, $\partial X = \bigcup_{i=1}^{m} D_i$, D_i = irred. comp.

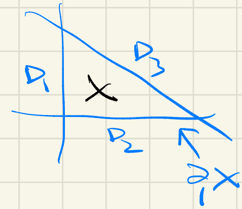
Defn the dual bdy complex $\mathbb{D}\partial X$:= a simplicial complex s.t.

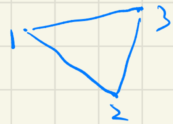
vertices i $\overset{1:1}{\longleftrightarrow}$ irreducible components D_i

Add a k-cell $[i_0 \cdots i_k]$ $\Longleftrightarrow$ $\bigcap_{j=0}^{k} D_{i_j} \neq \emptyset$.

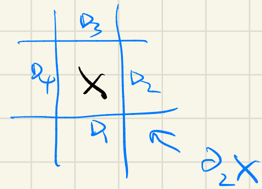
E.g. $X = (\mathbb{C}^*)^2$ $\rightsquigarrow$

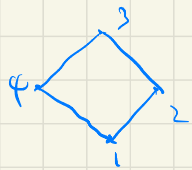
Log compactification 1: $X \hookrightarrow \mathbb{CP}^2$



$\Rightarrow$ $\mathbb{D}\partial_1 X =$ 

Log compactification 2: $X \hookrightarrow (\mathbb{CP}^1)^2$



$\Rightarrow$ $\mathbb{D}\partial_2 X =$ 

Properties of dual boundary complexes

1) (Danilov 75): the homotopy type of $\mathbb{D}\partial X$ is an invariant of X. (eg: see above)

2) (Payne ?): the homology of $\mathbb{D}\partial X$ captures information about the weight filtration on $H^*(X)$:

$$\tilde{H}_{i-1}(\mathbb{D}\partial X) \cong Gr^W_{2d} H^{2d-i}(X),$$

where $d = \dim_{\mathbb{C}} X$.

3) (Simpson 16): $Z = \mathbb{A}^1 \times Y \overset{cl}{\hookrightarrow} X \underset{\text{open dense}}{\hookleftarrow} U := X \setminus Z \Rightarrow \mathbb{D}\partial X \underset{\substack{\uparrow \\ \text{homotopy equivalent}}}{\simeq} \mathbb{D}\partial U$

As the 1st step of the geometric P=W conj, have:

Homotopy type conjecture (Katzarkov–Noll–Pandit–Simpson 15).

$\exists$ homotopy equivalence $\partial_\infty \partial M_B \simeq S^{d-1}$,

$d = \dim_{\mathbb{C}} M_B$.

What do we know?

Regular case:

. Komyo 13 : true for $G = SL_2(\mathbb{C})$, $C = \mathbb{CP}^1$ with 5 punctures.

. Simpson 16 : true for $G = SL_2(\mathbb{C})$, $C = \mathbb{CP}^1$ with k punctures $(k \geq 4)$

(Singular case:

. Mauri, Mazzon, Stevenson 18 : true for $G = GL_n, SL_n$, $g(C) = 1$)

Irregular case:

Main theorem in this talk (S 4):

the homotopy type conj holds for $M_B = M_1(\beta)$,

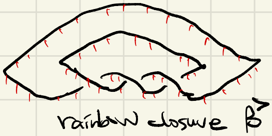
($G = GL_n$, $C = \mathbb{CP}^1$ with one puncture,

the irregular singularity is specified by

$\beta = \forall$ n-strand positive braid s.t. $\hat{\beta}$ is connected)

$\mathcal{M}_B = \mathcal{M}_1(\beta) = ?$ E.g.:

$\beta = \times\times\times \rightsquigarrow$



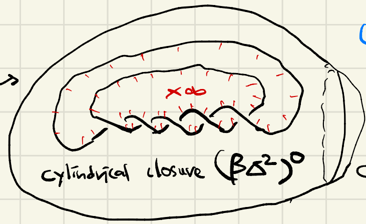
rainbow closure $\beta^>$ $\rightsquigarrow \mathcal{M}_1(\beta^>)$

Fact: $\cong$ (see 1st talk: "closure")

$\rightsquigarrow \mathcal{M}_B$.

$\beta\Delta^2 = \times\times\times\times\times$

Add a full twist $\rightsquigarrow$



cylindrical closure $(\beta\Delta^2)^\circ$ $\mathbb{CP}^1$

$\rightsquigarrow \mathcal{M}_1(\mathbb{CP}^1, \infty, (\beta\Delta^2)^\circ)$

The full geometric P=W conjecture (KNPS15):

$\exists$ homotopy commutative diagram

$$\begin{tikzcd} N^*_{Dol} \arrow[r, "\Phi"', "\sim", "\swarrow NAH"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & N^*_B \arrow[d, "q"] \\ (A \setminus B_R(0))/\mathbb{R}_+ = S^{d-1} \arrow[r, "\sim"] & \mathbb{D}\partial\mathcal{M}_B \end{tikzcd}$$

N^*_B = punctured nbhd of $\partial\mathcal{M}_B$ in $\mathcal{M}_B$.

"Hitchin map at ∞" h

"asymptotic behavior of $\mathcal{M}_B$ at ∞".

hmpy type conj

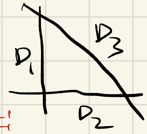
LHS:

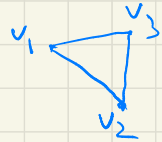
$\dim_{\mathbb{C}} = d$ even.

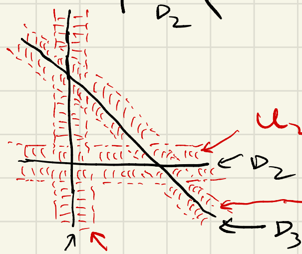
$$\begin{tikzcd} h^{-1}(B_R(0)) \arrow[r, hook] \arrow[d] & \mathcal{M}_{Dol} \arrow[d, "h", "\text{Hitchin map}"'] & N^*_{Dol} \arrow[l] \arrow[d] \arrow[drr, "\bar{h}"] & & \\ B_R(0) \arrow[r, hook] & \mathbb{A}^{\frac{d}{2}} & \mathbb{A} \setminus B_R(0) \arrow[l] \arrow[rr] & & (\mathbb{A} \setminus B_R(0))/\mathbb{R}_+ = S^{d-1} \end{tikzcd}$$

$(R \gg 1)$

radial scaling

RHS: E.g. $\partial M_B =$ 

$\Rightarrow \mathbb{D}\partial M_B =$ 

$\rightsquigarrow$ $N_B^* =$ 

$= U_1 \cup U_2 \cup U_3$.

(open cover)

$U_2 = \cdots$

$U_3 = \cdots$

D_1 $U_1 =$ punctured tubular nbhd of D_1 in M_B

Analysis $\Rightarrow$ $\exists$ a partition of unity $\{\rho_i\}_{i=1}^3$ associated to $\{U_i\}$

i.e. $\mathrm{supp}(\rho_i) \subset U_i$, $0 \le \rho_i \le 1$. & $\sum \rho_i \equiv 1$ on N_B^*

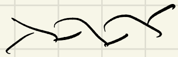
$$\rightsquigarrow \quad \lambda : N_B^* \longrightarrow \mathbb{D}\partial M_B$$
$$x \longmapsto \sum \rho_i(x) v_i$$

Fact: λ is well-defined up to homotopy

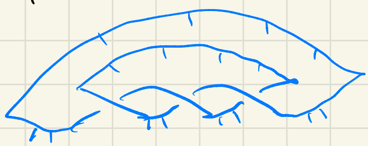
Expect: geometric $P=W$ recovers more information about the cohomological $P=W$ conjecture.

2. Legendrian knots and constructible sheaves

$\beta =$ n-strand positive braid

e.g.: $n=2$, $\beta = \sigma_1^3 =$ 

⇝ the rainbow closure $\beta^{>}$:

In e.g.: $\beta^{>} =$  $\hookrightarrow J^1\mathbb{R}_x = \mathbb{R}^3_{x,y,z}$

⇝ this defines a Legendrian link

$$\beta^{>} \hookrightarrow (J^1\mathbb{R}_x, \alpha) = (\mathbb{R}^3_{x,y,z}, \alpha = dz - y\,dx)$$

View $\beta^{>} \hookrightarrow J^1\mathbb{R}_x \xhookrightarrow{\text{open}} S^*(\mathbb{R}^2_{x,z})$ as a microsupport condition:

unit cotangent bdle

⇝ (Shende–Treumann–Zaslow)

"Defn": $\mathfrak{M}_B = \mathfrak{M}_1(\beta^{>}) :=$ the STZ moduli stack of objects of $\mathcal{C}_1(\beta^{>}; k) :=$ dg category of constructible sheaves $\mathcal{F}$ on $\mathbb{R}^2_{x,z}$ s.t.:

(1st talk) ("localized" at quasi-isom's).

- the microsupport at infinity $SS(\mathcal{F}) \cap S^*\mathbb{R}^2_{x,z} \subset \beta^{>}$,
- $\mathcal{F}$ has compact support.
- $\mathcal{F}$ is of microlocal rank 1: $\mu mon(\mathcal{F}) \simeq \underline{k} \in Loc(\beta^{>})$

microlocal monodromy.

= microlocal stalk "$\mathcal{F}|_{\beta^{>}}$"

up to a deg shift.

Defn: $\mathcal{M}_B = \mathcal{M}_1(\beta^{>})$ is the good moduli space associated to the STZ moduli stack $\mathfrak{M}_B = \mathfrak{M}_1(\beta^{>})$.

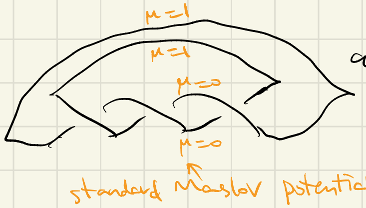
Good moduli spaces (Alper 13):

- A good moduli space of a reasonable (e.g. locally of finite presentation) algebraic stack, when exists, is unique.

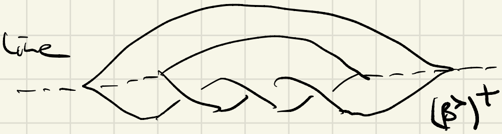
- X affine $\circlearrowleft$ G linear reductive algebraic
  $\Rightarrow$ the good moduli space of the quotient stack $[X/G]$ is $\operatorname{Spec} \mathcal{O}(X)^G$. ⊛

Combinatorial description:

$\beta^{>} =$



add dashed line $\rightsquigarrow$

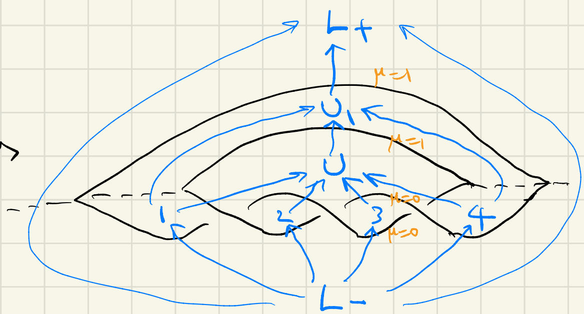


$(\beta^{>})^{+}$

induces a stratification $\mathcal{S}$ of $\mathbb{R}^2_{x,z}$, which is regular.

- 0-cells = cusps, crossings
- 1-cells = arcs = conn. comp's of $(\beta^{>})^{+} - \{\text{0-cells}\}$.
- 2-cells = regions = conn. comp's of $\mathbb{R}^2_{x,z} - \{\text{0-cells}, \text{1-cells}\}$

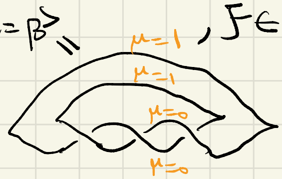
$\rightsquigarrow$

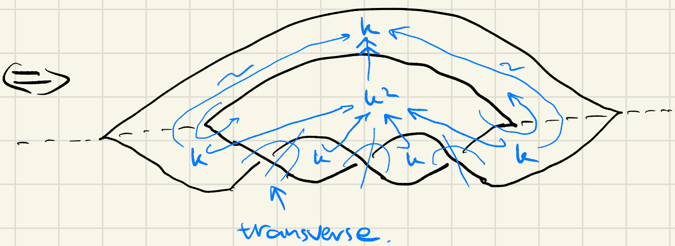


quiver with relations $Q = \overline{Q}_{\mathcal{S}}$

Recall Ex from the 1st talk:

$\Lambda = \beta^{>}$, $\mathcal{F} \in \mathcal{C}_1(\Lambda; k) \Leftrightarrow$

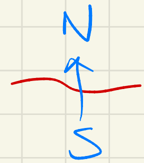


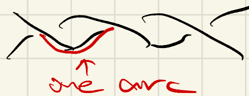


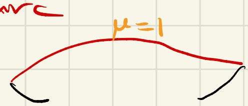
⇝ <u>In general</u>:

- Define a <u>dimension vector</u> $\vec{d}$, uniquely characterized by:

- $\vec{d}(L_\pm) = 0$.

- For each solid arc $\overset{\mu=0}{\sim}$ of β ⇝  $\Rightarrow d(N) = d(S)+1$

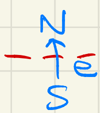
(e.g.:  β)

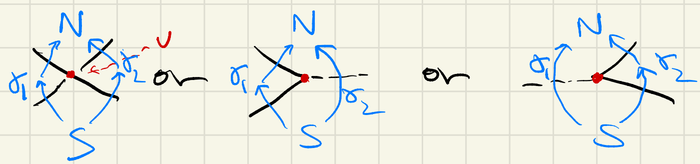
- For each arc  $\in \beta^>$

⇝  $\Rightarrow d(N) = d(S) - 1$

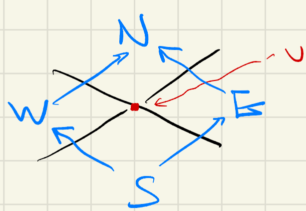
<u>Defn</u>: • $\operatorname{Rep}(Q, \vec{d}) := \prod_{e: R \to S} \operatorname{Hom}(k^{d(R)}, k^{d(S)}) = \mathbb{A}^N_k$.

- $\operatorname{Rep}_{\beta^>}(Q, \vec{d}) := \{ F \in \operatorname{Rep}(Q, \vec{d}) \text{ s.t. } ①, ②, ③, ④ \}$.

①:  dash arc $\Rightarrow F(e) : F(S) \xrightarrow{\sim} F(N)$.

② u = singularity 

$\Rightarrow F(\gamma_1) = F(\gamma_2)$

③ u = crossing 

$$\Rightarrow \operatorname{Tot}\left(\begin{tikzcd} & F(N) & \\ F(W) \arrow[ur] & & F(E) \arrow[ul] \\ & F(S) \arrow[ul] \arrow[ur] & \end{tikzcd}\right) \text{ is acyclic.}$$

④ $\forall$ solid arc  $\Rightarrow$ rank $F(e)$ is maximal.

(i.e. $F(e)$ is injective or surjective)

• $GL(\vec{d}) := \prod_{R \in Q_0} GL(d(R))$

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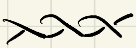
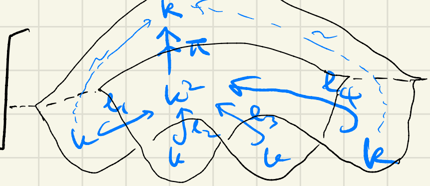
easy fact: $GL(\vec{d}) \curvearrowright Rep_{\beta^>}(Q, \vec{d})$.

<u>Not so trivial fact</u>: $\mathcal{M}_1(\beta^>) \cong \left[Rep_{\beta^>}(Q, \vec{d}) / GL(\vec{d}) \right]$

<u>In Ex</u>: $\beta =$ 

$Rep_{\beta^>}(Q, \vec{d}) = \Big\{$ 

s.t. $[\ell_i] \pitchfork [\ell_{i+1}]$, $\forall i \in \mathbb{Z}/5$

where $[\ell_i] := \mathrm{Im}(\ell_i)$, $i = 1,2,3,4$

$[\ell_0] := \mathrm{Ker}(\pi)$ $\Big\}$

$\circlearrowleft$

$GL(\vec{d}) = GL(2) \times GL(1)^5$

$$\Rightarrow \mathcal{M}_1(\beta^>) \cong \left[\left\{ [\ell_i] \in (\mathbb{P}^1(k))^5 : [\ell_i] \neq [\ell_{i+1}] \; \forall i \in \mathbb{Z}/5 \right\} \Big/ GL(2) \right]$$

<u>Issue</u>: In general, $Rep_{\beta^>}(Q, \vec{d})$ is <u>not</u> necessarily affine.

e.g. $\{\ell : k \hookrightarrow k^2\} = \mathbb{A}^2_k - \{0\}$.

(Want to apply ⊛)

<u>Sol</u>: $\exists$ an alternative perspective via contact geometry

3. Betti moduli space via augmentations

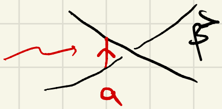
view $\beta^> \hookrightarrow J^1\mathbb{R}_x = \mathbb{R}^3_{x,y,z}$ as a Legendrian link in contact geometry.

SFT for $(V = \mathbb{R}^3_{x,y,z},\ \Lambda = \beta^>)$ $\rightsquigarrow$

contact mfld   Legendrian submfld.

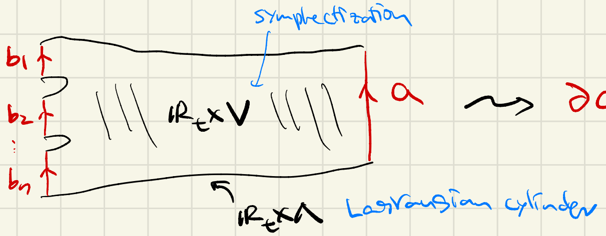
$\mathcal{A}(\beta^>) :=$ the Chekanov–Eliashberg DGA,

· generators = Reeb chords of $\Lambda = \beta^>$

(flow lines of $R_\alpha = \partial_z$ $\rightsquigarrow$ )

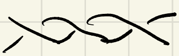
deg -1

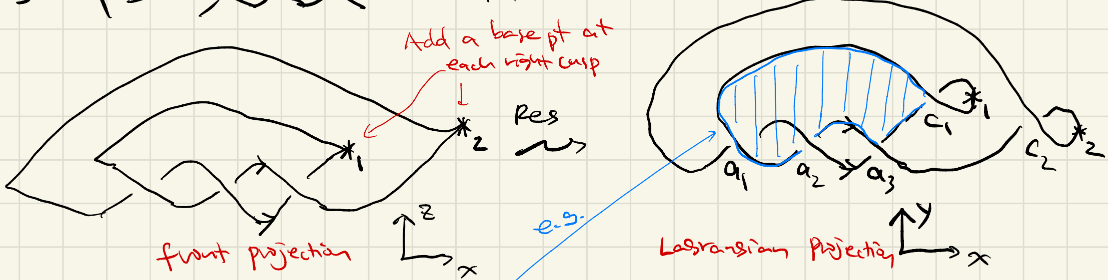
· differential = counts holomorphic disks



$\rightsquigarrow\ \partial a = \sum \varepsilon\, b_1 b_2 \cdots b_n$

weights

E.g. $\beta =$  $\rightsquigarrow$



$\Rightarrow \mathcal{A}(\beta^>) = \mathbb{Z}\langle t_1^{\pm 1}, t_2^{\pm 1}, a_1, a_2, a_3, c_1, c_2 \rangle$, $|t_i^{\pm 1}| = 0$, $|a_i| = 0$, $|c_i| = 1$.

$$\begin{cases} \partial c_1 = t_1^{-1} + a_1 + a_3 + a_1 a_2 a_3 \\ \partial c_2 = t_2^{-1} + a_2 + (1 + a_2 a_3)\, t_1 (1 + a_1 a_2) \end{cases}$$

$\partial t_i^{\pm 1} = 0$, $\partial a_i = 0$.

Augmentations:

Defn: the augmentation variety associated to $\Lambda = \beta^{>}$ is:

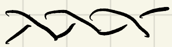
$$\mathrm{Aug}(\beta^{>}, \vec{*}; k) := \left\{ \varepsilon : \mathcal{A}(\beta^{>}) \longrightarrow (k, 0) \ \ \mathbb{Z}\text{-graded DGA maps} \right\}$$

"framed augmentation variety"

$$\mathrm{Aug}(\beta^{>}; k) := \left\{ \varepsilon \in \mathrm{Aug}(\beta^{>}, \vec{*}; k) \mid \varepsilon(t_i) = 1, \ 1 \le i < n \right\}$$

augmentation variety.

Affine

E.g.: $\beta =$  $\Longrightarrow$

$$\mathrm{Aug}(\beta^{>}, \vec{*}; k) \cong \left\{ (\varepsilon(t_1), \varepsilon(a_i)) \in k^{*} \times k^{3} \mid \varepsilon(t_1)^{-1} + \varepsilon(a_1) + \varepsilon(a_3) + \varepsilon(a_1 a_2 a_3) = 0 \right\}$$

$$\mathrm{Aug}(\beta^{>}; k) \cong \left\{ 1 + x + z + xyz = 0 \right\}. \quad (x, y, z) = \varepsilon(a_1, a_2, a_3).$$

Theorem (S 21). If $\beta^{>}$ is connected, then:

(1). $\exists$ natural isom of algebraic stacks:

$$\left[\mathrm{Aug}(\beta^{>}, \vec{*}; k) / \mathbb{G}_m^n \right] \cong \left[\mathrm{Rep}_{\beta^{>}}(Q, \vec{d}) / GL(\vec{d}) \right] \cong \mathcal{M}_1(\beta^{>})$$

$\exists$

(2) $\mathcal{M}_1(\beta^{>}) \underset{(\text{by } \circledast)}{\cong} \mathrm{Spec}\, \mathcal{O}\left(\mathrm{Aug}(\beta^{>}, \vec{*}; k)\right)^{\mathbb{G}_m^n} \cong \mathrm{Aug}(\beta^{>}; k)$ (say $k = \mathbb{C}$)

is smooth, connected affine variety.

(3) The homotopy type conjecture for $\mathcal{M}_B = \mathcal{M}_1(\beta^{>})$ holds:

$$\mathbb{D}\partial \mathcal{M}_B \simeq S^{d-1}, \quad \text{where } d = \dim_{\mathbb{C}} \mathcal{M}_B.$$

Idea of proof:

Ng–Rutherford–Shende–Sivek–Zaslow

(1): inspired by "Augmentations are sheaves" (NRSSZ 20)

(2): $\beta^>$ is connected $\Rightarrow$ $\mathbb{G}_m^n/\mathbb{G}_m$ acts freely on $\mathrm{Aug}(\beta,*;k)$.

(3): $\exists$ a natural cell decomposition (well known)

$$\mathrm{Aug}(\beta^>;k) = \bigsqcup_{\rho\in NR(\beta^>)} \mathrm{Aug}^{\rho}(\beta^>;k), \quad \mathrm{Aug}^{\rho}(\beta^>;k) \cong (k^*)^{s(\rho)-n+1}\times k^{r(\rho)}$$

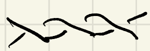
s.t. • $\exists!\ \rho_m$ s.t. $r(\rho_m)=0$.

• $(k^*)^{s(\rho_m)-n+1} \hookrightarrow \mathrm{Aug}(\beta^>;k)$ is open and dense.

(by fact, properties of dual boundary complexes: $Z=\mathbb{A}^1\times Y \xrightarrow{cl} X \Rightarrow \mathbb{D}\partial X \simeq \mathbb{D}\partial(X\setminus Z)$).

$$\Rightarrow \underbrace{\mathbb{D}\partial(\mathbb{C}^*)^{\overset{s(\rho_m)-n+1}{=d}}}_{= S^{d-1}} \simeq \underbrace{\mathbb{D}\partial\,\mathrm{Aug}(\beta^>;\mathbb{C})}_{= \mathbb{D}\partial M_B}$$

$\square$

E.g.: $\beta =$  $\Rightarrow$

$$m_1(\beta^>) \cong \left[\left\{ [l_i]\in(\mathbb{P}^1(k))^5 : [l_i]\neq[l_{i+1}],\ i\in\mathbb{Z}/5 \right\}/GL_2\right]$$

$$\cong \left[\left\{ [l_0]=[0:1],\ [l_1]=[1:0],\ [l_i]\neq[l_{i+1}] \right\}/\mathbb{G}_m^2\right]$$

$$\cong \left[\left\{ x+z+xyz\neq 0 \right\}/\mathbb{G}_m^2\right] \Rightarrow M_1(\beta)\cong\{1+x+z+xyz=0\}$$

$$\cong \mathrm{Aug}(\beta^>;k)$$

$[l_2] = [(0,1) - x(1,0)] = [-x:1]$

$[l_3] = [(1,0) - y(-x,1)] = [1+xy:-y]$

$[l_4] = [(-x,1) - z(1+xy,-y)] = [-x-z-xyz : 1+yz]$

$\neq [l_0]=[0:1]$

Cell decomposition :

$$M_B = \mathrm{Aug}(\vec{\beta};k) = \mathrm{Aug}^{\rho_1}(\vec{\beta};k) \sqcup \mathrm{Aug}^{\rho_2}(\vec{\beta};k) \sqcup \mathrm{Aug}^{\rho_3}(\vec{\beta};k)$$

$$\mathrm{Aug}(\vec{\beta};k) = \{1+x+z+xyz=0\}$$

$$\mathrm{Aug}^{\rho_1}(\vec{\beta};k) = \left\{ \begin{matrix} x\neq 0,\ 1+xy=0 \\ 1+x+z+xyz=0 \end{matrix} \right\} \cong k$$

$$\mathrm{Aug}^{\rho_2}(\vec{\beta};k) = \left\{ \begin{matrix} x=0,\ 1+xy\neq 0 \\ 1+x+z+xyz=0 \end{matrix} \right\} \cong k$$

$$\mathrm{Aug}^{\rho_3}(\vec{\beta};k) = \left\{ \begin{matrix} x\neq 0,\ 1+xy\neq 0 \\ 1+x+z+xyz=0 \end{matrix} \right\} \cong (k^*)^2$$

$$\Rightarrow \quad \mathbb{D}\partial M_B \simeq \mathbb{D}\partial(\mathbb{C}^*)^2 = S^1$$

$\square$