Dual boundary complexes of Betti moduli spaces associated to Legendrian knots

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Outline

1 Htpy type conj.

2 Moduli of constructible sheaves

3 Betti moduli spaces ass. to Legendrian knots

4 (Prove main thm via augmentations)

Dual boundary complexes

- X: smooth quasi-projective variety over $\mathbb{C}$.
  Log compactification: smooth projective $\overline{X}$ with snc boundary divisor $\partial X = \overline{X} \setminus X$.
- Resolution of singularities$\rightsquigarrow \exists \overline{X}$.
  May also assume: the irreducible components of ∂X have *connected* intersections.
- Dual boundary complex $\mathbb{D}\partial X$: a simplicial complex s.t.:
  • vertices $\leftrightarrow$ irreducible components of ∂X;
  • k vertices spans a $(k-1)$-simplex iff the corresponding components have non-empty intersection.

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Example: $X = (\mathbb{C}^*)^2$

- Log compactification 1: $X \hookrightarrow X_1 = \mathbb{C}P^2_{[x:y:z]}$ with $\partial_1 X = X_1 \setminus X = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$.
  $\Rightarrow \mathbb{D}\partial_1 X =$ triangle $(\simeq S^1)$.
- Log compactification 2: $X \hookrightarrow X_2 = \mathbb{C}P^1_{[a:b]} \times \mathbb{C}P^1_{[c:d]}$ with $\partial_2 X = X_2 \setminus X = \{a = 0\} \cup \{b = 0\} \cup \{c = 0\} \cup \{d = 0\}$.
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Properties

- E.g.: $\mathbb{D}\partial\mathbb{C}^* \sim S^0$; $\mathbb{D}\partial\mathbb{C} \sim pt$; $\mathbb{D}\partial(X \times Y) \sim \mathbb{D}\partial X * \mathbb{D}\partial Y$. $\rightsquigarrow$ $\mathbb{D}\partial(\mathbb{C}^*)^d \sim S^{d-1}$, and $\mathbb{D}\partial(\mathbb{C} \times Y) \sim pt$.
- Danilov 75: The homotopy type of $\mathbb{D}\partial X$ is an invariant of X.
- Hacking 08, Payne 13?: The reduced homology of $\mathbb{D}\partial X$ captures information of the weight filtration on $H^\bullet(X)$:

$$\tilde{H}_{i-1}(\mathbb{D}\partial X) \simeq \mathrm{Gr}^W_{2d} H^{2d-i}(X),$$

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  G: reductive algebraic group over $\mathbb{C}$.
- $\mathcal{M}_B = \mathcal{M}_B(C, G; \ldots)$: (smooth) Betti moduli space (or character variety) of G-local systems over C.

Homotopy type conjecture (Katzarkov-Noll-Pandit-Simpson15)

$\exists$ *homotopy equivalence* $\mathbb{D}\partial\mathcal{M}_B \sim S^{d-1}$, $d = \dim_{\mathbb{C}}\mathcal{M}_B$.

- Simpson 16: true for $G = SL_2(\mathbb{C})$ and $C = P^1 \setminus \{y_1, \ldots, y_k\}$.
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Preview for main result

- $C = \mathbb{C}P^1$ with one puncture ∞. $G = GL_n(\mathbb{C})$.
- β: n-strand positive braid $\rightsquigarrow$ specifies the *irregularity singularity type* at ∞ by a *Stokes Legendrian knot* $(\beta\Delta^2)^\circ$.
- $\mathcal{M}_B := \mathcal{M}_1(\mathbb{C}P^1 \setminus \{\infty\}, (\beta\Delta^2)^\circ)$: associated Betti moduli space/wild character variety.

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Relation with P=W in nonabelian Hodge

- $\mathcal{M}_{Dol} = \mathcal{M}_{Dol}(C, G; \ldots)$: Dolbeault moduli space of (stable) G-Higgs bundles over C. Hitchin map: $h : \mathcal{M}_{Dol} \to \mathbb{A}$.
- Nonabelian Hodge (NAH) $\rightsquigarrow$ diffemorphism $\phi : \mathcal{M}_{Dol} \simeq \mathcal{M}_B$ $\Rightarrow \phi^* : H^\bullet(\mathcal{M}_B) \simeq H^\bullet(\mathcal{M}_{Dol})$.

P=W conjecture (de Cataldo-Hausel-Migliorini 12)

NAH identifies the weight filtration (algebraic geometry) on $H^*(\mathcal{M}_B)$ with the perverse-Leray filtration (topology) on $H^*(\mathcal{M}_{Dol})$ associated to h:

$$\phi^*(W_{2k} = W_{2k+1}) = P_k.$$

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- Top row: NAH near infinity; Bottom row: htyp type conj.
- $\bar{h}$: Hitchin map near infinity; α: "asymptotic behavior of $\mathcal{M}_B$ near infinity".
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Relation with log Calabi-Yau varieties

- Expect: all smooth Betti moduli spaces are log CY, i.e. $\exists$ log compactification with anticanonical boundary divisor.

Conjecture (Kontsevich)

The dual boundary complex of any log Calabi-Yau variety is (homotopy equivalent to) a sphere.

- Example: $X = E \times \mathbb{C}^* \hookrightarrow \overline{X} = E \times \mathbb{C}P^1$ with $\partial X = E \times \{0\} + E \times \{\infty\} = -K_{\overline{X}}$. We see $\mathbb{D}\partial X = S^0$.
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Constructible sheaves

- M: real analytic manifold. (our case: $M = C$)
  $\mathbb{K}$: base field (or commutative ring).
- $\mathcal{F}$: *constructible sheaf* on M. I.e. $\mathcal{F}$ is a cochain complex of sheaves of $\mathbb{K}$-modules over M s.t.
  - $\exists$ (nice) stratification $\mathcal{S} = \{S_\alpha\}$ of M s.t. $\mathcal{H}^\bullet(\mathcal{F})|_{S_\alpha}$ is locally constant, $\forall \alpha$.
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Micro-support

- Slogan: "Micro-support captures the codirections along which the local sections fail to propogate".

Definition

Let $x \in M$, and $\xi \in T^*_x M$. Say, $\xi \neq 0$. If $\exists$ a small ball $B_\epsilon(x) \subset M$ and a smooth function $f : B_\epsilon(x) \to \mathbb{R}$ with $f(x) = 0$, $df(x) = \xi$, s.t.

$$R\Gamma(f^{-1}(-\infty, \delta); \mathcal{F}) \xrightarrow{r} R\Gamma(f^{-1}(-\infty, -\delta); \mathcal{F}) \tag{1}$$

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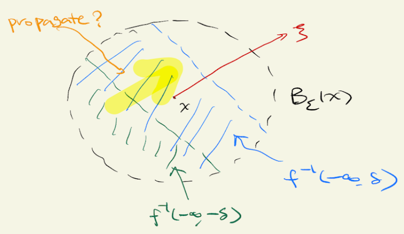

Figure: Propagation of local sections.

Definition (informal)

The *micro-support* (or *singular support*) of $\mathcal{F}$ is

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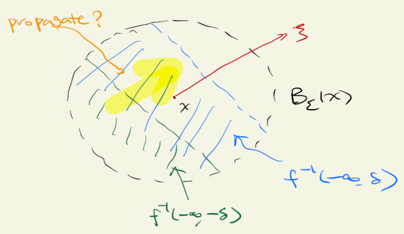

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Sheaf category

- Fact: $\mathrm{SS}(\mathcal{F}) \subset T^*M$ is a *conic, closed, (singular) Lagrangian*. $\rightsquigarrow \mathrm{SS}^\infty(\mathcal{F}) := \mathrm{SS}(\mathcal{F}) \cap S^*M$ (*micro-support at infinity*) is a *closed (singular) Legendrian subset*, i.e. $\sum p_i dq_i|_{(\mathrm{SS}^\infty(\mathcal{F}))^{sm}} = 0$.

Definition

$\Lambda \subset S^*M$ smooth Legendrian. $\mathcal{S}h_\Lambda(M;\mathbb{K})$: dg cat. of constructible sheaves $\mathcal{F}$ on M s.t. $\mathrm{SS}^\infty(\mathcal{F}) \subset \Lambda$, localized at quasi-iso's.

Theorem (Guillermou-Kashiwara-Shapira 12)

The dg category $\mathcal{S}h_\Lambda(M;\mathbb{K})$ is a Legendrian isotopy invariant of Λ.

- Case of interest: $M = C$ (Riemann surface), $\Lambda \subset S^*C$: Legendrian link. $\rightsquigarrow$ computable Legenrian knot invariants.

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Microlocal stalks and microlocal monodromy

- Slogan: "Microlocal stalk = the obstruction of the propagation of the local sections along a given (singular) covector".

Definition/Proposition

$\mathcal{F} \in Sh_\Lambda(M; \mathbb{K})$, then $\mathcal{F}|_{(x,\xi)} := \mathrm{Cone}(r)$ in (1) is independent of the choices of ϵ, f, δ, called the *microlocal stalk* of $\mathcal{F}$ at (x, ξ).

Definition/Proposition

$\exists$ a natural dg functor (*microlocal monodromy*)

$$\mu\mathrm{mon} : Sh_\Lambda(M; \mathbb{K}) \to \mathcal{L}oc(\Lambda; \mathbb{K}) \quad \text{(local systems on } \Lambda)$$

s.t. $\mu\mathrm{mon}(\mathcal{F})_{(x,\xi)} = \mathcal{F}|_{(x,\xi)}[-\mu(x,\xi)]$ for a Maslov potential μ.

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- The *STZ category* $\mathcal{C}_1(M, \Lambda; \mathbb{K})$ is the full dg subcategory of $\mathcal{S}h_\Lambda(M; \mathbb{K})$ consisting of $\mathcal{F}$ s.t. $\mathcal{F}$ has compact support, and $\mu\mathrm{mon}(\mathcal{F}) \simeq \underline{\mathbb{K}}$ is the trivial rank 1 local system on Λ.
- The *STZ moduli stack* $\mathfrak{M}_1(M, \Lambda; \mathbb{K})$ is the (underived) moduli stack of objects in $\mathcal{C}_1(M, \Lambda; \mathbb{K})$.
- The *STZ moduli space* $\mathcal{M}_1(M, \Lambda; \mathbb{C})$ is the *good moduli space* [Alper 13] of $\mathfrak{M}_1(M, \Lambda; \mathbb{C})$.

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Legendrian knots via braid closures

β: n-strand positive braid $\rightsquigarrow$ 2 closures:

- *Rainbow closure*: $\beta^{>} \hookrightarrow J^1\mathbb{R}_x \hookrightarrow S^*M$ with $M = \mathbb{R}^2_{x,z}$.
- *Cylindrical closure*: $(\beta\Delta^2)^\circ \hookrightarrow S^*M$ with $M = \mathbb{C}P^1 \setminus \{\infty\}$.

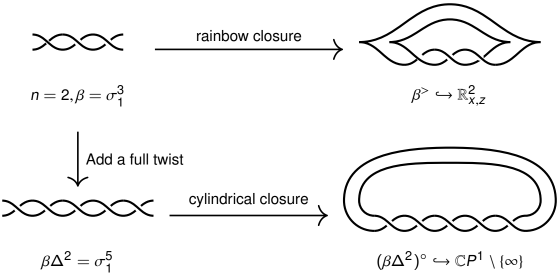

Figure: Braid closure: $n = 2, \beta = \sigma_1^3$.

Betti moduli spaces associated to Legendrian knots

Definition/Proposition

$\exists$ natural isomorphism $\mathcal{M}_1(\mathbb{R}^2_{x,z}, \beta^>; \mathbb{C}) \simeq \mathcal{M}_1(\mathbb{C}P^1 \setminus \{\infty\}, (\beta\Delta^2)^\circ; \mathbb{C})$. Denote either by $\mathcal{M}_1(\beta)$ (*Betti moduli space* in main theorem).

- Irregular Riemann-Hilbert $\rightsquigarrow \mathcal{M}_1(\beta)$ can be viewed as a Betti moduli space/wild character variety associated to $C = \mathbb{C}P^1$ with one irregular singularity at ∞, and $G = GL_n(\mathbb{C})$.
- Recall main theorem: the homotopy type conjecture holds for $\mathcal{M}_B = \mathcal{M}_1(\beta)$: $\mathbb{D}\partial\mathcal{M}_B \sim S^{\dim - 1}$.

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Combinatorial description via example: $n = 2, \beta = \sigma_1^3$

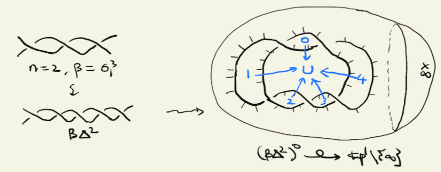

Figure: Braid closure: Q is the associated quiver (blue).

Fact: $\mathcal{F} \in C_1(\mathbb{C}P^1 \setminus \{\infty\}, (\beta\Delta^2)^\circ; \mathbb{K}) \Leftrightarrow F \in \mathrm{Rep}(Q)$ s.t.

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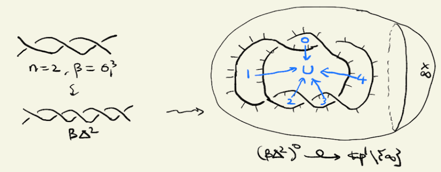

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- Symplectic field theory: Legendrian knot $\Lambda = \beta^{>} \hookrightarrow J^1\mathbb{R}_x \rightsquigarrow$ Legendrian contact homology DGA $\mathcal{A}(\Lambda, \vec{*}) \rightsquigarrow$ augmentation variety $\mathrm{Aug}(\Lambda, \vec{*}; \mathbb{K})$.
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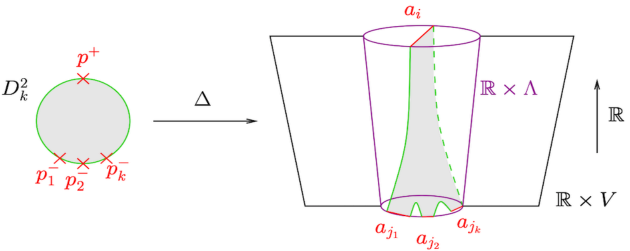
Legendrian Contact Homology DGAs

- SFT: contact pair $(V, \Lambda) \mapsto$ LCH DGA $\mathcal{A}(V, \Lambda)$ with:
  generators: Reeb chords of Λ;
  differential: counts holomorphic disks in the symplectization $\mathbb{R} \times V$, w/ boundary along the Lagrangian cylinder $\mathbb{R} \times \Lambda$ and meeting the Reeb chords at infinity. (see figure below)



LCH DGAs

Our case: $V = J^1\mathbb{R}_x$, Λ: Legendrian knot, $\mathcal{A}(\Lambda) = \mathcal{A}(\mathbb{R}^3, \Lambda)$.

- Chekanov-Eliashberg: combinatorial description of $\mathcal{A}(\Lambda)$.

Example ($\Lambda = \beta^{>}, \beta = \sigma_1^3$ and $n = 2$)

See Figure below, can compute:
$\mathcal{A}(\Lambda, \overrightarrow{*}) = \mathbb{Z}\langle t_1^{\pm 1}, t_2^{\pm 1}, a_1, a_2, a_3, c_1, c_2\rangle$, $|t_i| = 0 = |a_i|, |c_i| = 1$, and

$$\begin{aligned}
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$\mathcal{A}(\Lambda) = \mathcal{A}(\Lambda, \overrightarrow{*})|_{t_1=1, t_2=t}$.

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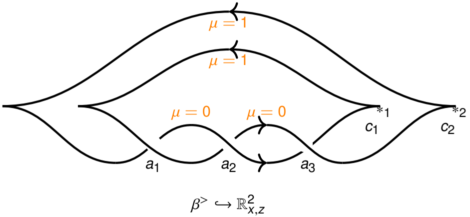

Figure: $\Lambda = \beta^{>} =$ the right-handed Legendrian trefoil knot.

- Additional data: place a base point $*_i$ at each right cusp c_i; Maslov potential $\mu \rightsquigarrow \mathcal{A}(\Lambda, \vec{*})$.
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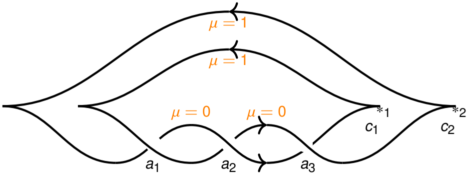

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The *augmentation variety* of $\mathcal{A}(\Lambda, \vec{*})$ is
$\mathrm{Aug}(\Lambda, \vec{*}; \mathbb{K}) := \{\epsilon \mid \epsilon : \mathcal{A}(\Lambda, \vec{*}) \to \mathbb{K}$ is a graded DGA map$\}$.
Similarly: $\mathcal{A}(\Lambda) \rightsquigarrow \mathrm{Aug}(\Lambda; \mathbb{K})$.

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Augmentations vs sheaves via example: $n = 2, \beta = \sigma_1^3$

- $\exists$ natural action of $T = \mathbb{G}_m^2$ on $\mathrm{Aug}(\beta^{>}, \vec{*}; \mathbb{K})$: $(\vec{\lambda} \cdot \epsilon)(t_1, a_2) = \lambda_2^{-1}\epsilon(t_1, a_2)\lambda_1$, $(\vec{\lambda} \cdot \epsilon)(t_2, a_1, a_3) = \lambda_1^{-1}\epsilon(t_2, a_1, a_3)\lambda_2$, $\forall \vec{\lambda} = (\lambda_1, \lambda_2) \in T, \epsilon \in \mathrm{Aug}(\Lambda, \vec{*}; \mathbb{K})$.
- Recall $\mathfrak{M}_1(\beta) = [\{[\ell_i] \in \mathbb{P}^1(\mathbb{K})^5 | [\ell_i] \neq [\ell_{i+1}], \forall i \in \mathbb{Z}/5\}/GL_2(\mathbb{K})] \Rightarrow$

$$\begin{aligned} \mathfrak{M}_1(\beta) &\cong [\{[\ell_0] = [0:1], [\ell_1] = [1:0], [\ell_i] \neq [\ell_{i+1}]\}/\mathbb{G}_m^2] \\ &\cong [\{x + z + xyz \neq 0\}/\mathbb{G}_m^2]^1 \\ &\cong [\mathrm{Aug}(\beta^{>}, \vec{*}; \mathbb{K})/T]^2 \\ &\cong [\{1 + x + z + xyz = 0\}/\mathbb{G}_m] \ (\mathbb{G}_m \text{ acts trivially}) \end{aligned}$$

[1] $[\ell_2] = [-x : 1], [\ell_3] = [(1,0) - y(-x,1)] = [1 + xy : -y], [\ell_4] = [(-x,1) - z(1 + xy, -y)] = [-x - z - xyz : 1 + yz] \neq [\ell_0] = [0 : 1]$.

[2] $\epsilon(a_1, a_2, a_3) = (x, y, z)$.

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Cell decomposition via example: $n = 2, \beta = \sigma_1^3$

- $\Rightarrow$ Good moduli space:
  $\mathcal{M}_1(\beta) \cong \{1 + x + z + xyz = 0\} \cong \mathrm{Aug}(\beta^>; \mathbb{C})$, $d = \dim = 2$.
- Cell decomposition:

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- Dual boundary complexes:
  $\mathbb{D}\partial\mathcal{M}_1(\beta) = \mathbb{D}\partial\mathrm{Aug}(\beta^>; \mathbb{C}) \sim {}^3\mathbb{D}\partial\mathrm{Aug}^{\rho_m}(\beta^>; \mathbb{C}) = \mathbb{D}\partial(\mathbb{C}^*)^2 = S^1$.

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[3] Simp 16: $U \subset X$ open dense, $Z := X \setminus U \cong \mathbb{A}^1 \times Y$, then $\mathbb{D}\partial X \sim \mathbb{D}\partial U$.

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Thanks!